A Brief Intro to Synthetic Spectra

合成的(?)

Summary

$$\begin{tikzcd}
 & Sp \arrow[ddl, bend right, "Id"'] \arrow[d, "\nu"] \arrow[ddrr, bend left, "E_*(-)"] & & \\
 & Syn_E \arrow[dl, "(-)[\tau^{-1}]"'] \arrow[dr, "C\tau\otimes-"] & & \\
Sp & & Mod_{C\tau} \arrow[r, hook] & Stable_{E_*E}
\end{tikzcd}$$

Goal Understand this diagram!

Reference Burklund, Subramanian, arXiv 2403.06724

Pstrągowski. arXiv 1803.01804.

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1. Motivations.

① **Q** When can : $X \longrightarrow X^{tC_p}$ w.e. ?

A1 Sp.

A2 X bdd below + p-complete + H_*X is I-complete

$\Rightarrow$ ✓.

IDEA : "can" induces iso on E_2-pages of corres. Adams SS.

RHS needs skeleta filtration + Adams SS

then take lim.

Very direct in Synthetic Spectra!

② Motivic $C\tau$ - method

Use to tackle the last Kervaire problem.

$$\tau\text{-Bockstein SS} \xleftrightarrow{\cdots} \text{ANSS}$$

2. Synthetic Spectra.

Slogan $\cdots = \mathrm{Sp}$ + assoc. Adams SS

encode info π_* + Adams SS.

Convention presentable . stable . sym. mon. ∞-cat.

Def $P \in \mathrm{Sp}$ is called finite E-projective . if

1) $P \in \mathrm{Sp}^{\omega}$

2) $E_* P$ proj as E_*-mod & f.g.

$\mathrm{Sp}_E^{fp} \subseteq \mathrm{Sp}$: all such spectra.

Here E satisfies :

1) A_∞ - ring / E_1 - ring.

2) Adams type $\begin{cases} E \simeq \operatorname{colim}_n E_n \\ E_n \in \mathrm{Sp}_E^{fp} . \\ E^* E_n \simeq \mathrm{Hom}_{E_*} (E_* E_n . E_*). \end{cases}$

e.g. $H\mathbb{Z}$, $H\mathbb{F}_p$. $K(n)$.

Landweber exact functors (e.g. MU).

Def Product-preserving sheaf of spectra

$$X(c \amalg c') \simeq X(c) \amalg X(c')$$

Def E-based synthetic spectra $\mathrm{Syn}_E = \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_E^{fp})$.

product-pre. sheaf of sp.

It comes w/ a functor (lax sym. mon.)

$$\nu : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$$

called "synthetic analogue".

connective cover

Construction $\nu : \mathrm{Sp} \xrightarrow{y} \mathrm{Shv}_\Sigma(\mathrm{Sp}_E^{fp}) \xrightarrow{\Sigma^\infty_+} \mathrm{Syn}_E$

y Yoneda emb. $X \longmapsto \mathrm{Map}_{\mathrm{Sp}_E^{fp}}(-, X)$.

Properties 1) ν preserves filtered colims.

fully faithful. additive.

2) $X \xrightarrow{f} Y \xrightarrow{g} Z$ cofib seq. then

$\nu X \xrightarrow{\nu f} \nu Y \xrightarrow{\nu g} \nu Z$ cofib seq

$$\Updownarrow$$

$E_*X \xrightarrow{f_*} E_*Y \xrightarrow{g_*} E_*Z$ s.e.s.

Def Bigraded sphere $S^{\overset{\mathbb{Z}}{t,w}} = \Sigma^{t-w}\nu S^w$.

Here $S^w = \Sigma^w \mathbb{S}$

Htpy of synthetic sp : $\pi_{t,w}X = [S^{t,w}, X]$.

Lem $P \in \mathrm{Sp}_E^{fp}$. $X \in \mathrm{Syn}_E$. Then

1) $\mathrm{Map}(\nu P, X) \simeq \Omega^\infty X(P)$.

2) $\pi_{t,w} X \simeq \pi_{t-w} X(S^w)$.

pf. $\mathrm{LHS} \simeq \mathrm{Map}(\Sigma^\infty_+ \mathrm{Map}(-,P), X)$

$\simeq \mathrm{Map}(\mathrm{Map}(-,P), \Omega^\infty X)$

$\simeq \mathrm{RHS}$.

Cor $X \in \mathrm{Sp}$. $\pi_{t,w} \nu X = \pi_t X$ when $t-w \geq 0$.

pf. $\pi_{t,w} \nu X = \pi_{t-w} \nu X(S^w)$

$= \pi_{t-w} \mathrm{Map}(S^w, X)$

$= \pi_t X$.

- "τ" map.

Def $\tau : \underset{S^{-1,-1}}{\nu S^{-1}} \longrightarrow \underset{S^{-1,0}}{\Omega(\nu S^0)}$ induced by $\Omega S^0 \simeq S^{-1}$.

$\tau \in \pi_{-1,-1} S^{-1,0} = \underset{[S^{0,-1}, S^{0,0}] = [\Sigma \nu S^{-1}, \nu S^0]}{\underset{\parallel}{\pi_{0,-1} S^{0,0}}}$

Prop $\forall\ P \in \mathrm{Sp}_E^{fp}$. $X \in \mathrm{Syn}_E$

1) $(\underbrace{\Sigma^{t,w} X}_{X \otimes S^{t,w}})(P) = \Sigma^{t-w} X(\Sigma^{-w} P)$.

2) commutative diagram:

$$\begin{tikzcd}
(\Sigma^{-1,-1} X)(P) \arrow[r, "\tau \otimes X"] \arrow[d, "\simeq"] & (\Sigma^{-1,0} X)(P) \arrow[d, "\simeq"] \\
{} & {}
\end{tikzcd}$$

$$X(\Sigma P) \xrightarrow{\phi} \Omega X(P)$$

ϕ is called "colim - to - lim comparsion".

$\Longrightarrow \quad \tau \otimes X \simeq \phi.$

pf. 1) $\text{LHS} = S^{t,w} \otimes X = \Sigma^{t-w} \nu S^{w} \otimes X$

$= \Sigma^{t-w}(\nu S^{w} \otimes X) = \Sigma^{t-w}(\Sigma^{w,w} X).$

FACT (Piotr Lem 2.27)

$$\begin{tikzcd} \overset{\mathrm{Map}(-,c)}{\overset{=}{y(c)}} \otimes - \arrow[d, phantom, "\wr"] : & \mathrm{PSh}(\mathcal{C}) \arrow[r] & \mathrm{PSh}(\mathcal{C}) \\ - \otimes c^{*} : & \mathcal{C} \arrow[r] & \mathcal{C}. \end{tikzcd}$$

"excellent ∞-site w/ sym. mon. str + blah blah."

$\Sigma^{w,w} X = \nu S^{w} \otimes X = y(S^{w}) \otimes X \simeq X \otimes S^{-w}$

Rest omitted.

2) by (1).

Question $\tau_{<0} S^{0,0}$.

- $C_\tau = \mathrm{cofib}\, \tau \simeq \tau_{\leq 0} S^{0,0}$. has an E_∞-alg str / $S^{0,0}$.

Similarly $\Sigma^{0,-1} \nu X \xrightarrow{\tau \otimes \nu X} \nu X \longrightarrow C_\tau \otimes \nu X$. (*)

$\rightsquigarrow \quad C_\tau \otimes \nu X \simeq \tau_{\leq 0}(\nu X)$

$\forall\, X \in \mathrm{Syn}_E$, can find $\{X_n\} \in \mathrm{Syn}_E$ s.t.

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r, "\tau \otimes X_1"] \arrow[d] & X_1 \arrow[r, "\tau \otimes X"] \arrow[d] & X_0 = X \arrow[d] \\ & C_\tau \otimes X_2 & C_\tau \otimes X_1 & C_\tau \otimes X_0 \end{tikzcd}$$

induced by (*).

- τ-invertible synthetic spectra.

Def $X \in \mathrm{Syn}_E$ is τ-invertible if $\tau : \Sigma^{0,-1}X \xrightarrow{\simeq} X$

Write $\mathrm{Syn}_E(\tau^{-1})$ = all such things

FACT

$$\begin{tikzcd} \mathrm{Syn}_E(\tau^{-1}) \arrow[r, hook, shift left=1.5, "\mathrm{incl}"] \arrow[r, phantom, "\perp"] & \mathrm{Syn}_E \arrow[l, shift left=1.5, "\tau^{-1}"] \end{tikzcd}$$

$$\tau^{-1}X := \mathrm{colim}\,(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \longrightarrow \cdots)$$

$$\begin{aligned} \mathrm{Map}(\tau^{-1}X, Y) &= \mathrm{Map}(\mathrm{colim}_n\, \Sigma^{0,n}X, Y) \\ &= \lim_n \mathrm{Map}(X, \Sigma^{0,-n}Y) \\ &= \lim_n \mathrm{Map}(X, Y) = \mathrm{Map}(X, Y). \end{aligned}$$

Let $Y(X)$:= presheaf of spectra

$\forall\, P \in \mathrm{Sp}_E^{fp}$. $Y(X)(P) = F(P, X)$

Note $\nu X = F(P, X)_{\geq 0}$.

Prop 1) $\nu X \longrightarrow Y(X)$ τ-inversion, i.e.

$$\tau^{-1}\nu X \simeq Y(X)$$

2) $Y(X)_{\geq 0} \simeq \nu X$.

pf.

$$\begin{tikzcd} \nu X \arrow[r] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & Y(X) \arrow[r] & C \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\ \mathrm{Syn}_{\geq 0} & \Rightarrow & \mathrm{Syn}_{\leq -1} \end{tikzcd}$$

$\Rightarrow$ (2) ✓.

For (1). recall $(\tau \otimes X) \simeq (X(\Sigma P) \longrightarrow \Omega X(P))$. $^{(*)}$

$$\begin{tikzcd} Y(X)(\Sigma P) = F(\Sigma P, X) \arrow[d, "(*)\ \tau"] \\ {} \end{tikzcd}$$

$$\Omega F(P,X) \simeq \Omega Y(X)(P)$$

$$\Sigma^{0,-1} Y(X)(P) = \Sigma Y(X)(\Sigma P) \simeq Y(X)(P).$$

$$\Rightarrow \quad Y \in \mathrm{Syn}_E(\tau^{-1}).$$

Taking τ^{-1} into

$$\begin{tikzcd} \tau^{-1}\nu X \arrow[r] & \tau^{-1}Y(X) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \tau^{-1}C \\ & Y(X) & \end{tikzcd}$$

Suffice to show $\tau^{-1}C \simeq 0$. ✓

FACT (Lem 4.35) $X \in \mathrm{Syn}_E$. k-coconnective for some $k \in \mathbb{Z}$.

then $\tau^{-1}X \simeq 0$.

Thm (4.37) Y fully faithful. lax sym mon.

$$\mathrm{Sp} \xrightarrow[\simeq]{Y} \mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Mod}_{\tau^{-1}S^{0,0}}(\mathrm{Syn}_E)$$

modules in Syn_E / $\tau^{-1}S^{0,}$

Cor $\nu : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$ fully faithful emb.

Def Underlying functor

$X \in \mathrm{Syn}_E$. $\tau^{-1}X = \mathrm{colim}_n \Sigma^{-n} X(S^{-n})$

Properties 1)

$$\begin{tikzcd} \mathrm{Syn}_E \arrow[r, shift left, "\tau^{-1}"] \arrow[r, phantom, "\perp"] & \mathrm{Sp} \arrow[l, shift left, "Y"] \end{tikzcd}$$

2) $Z \in \mathrm{Sp}$. $\tau^{-1}\nu Z \simeq Z$

3) $\tau^{-1}S^{t,w} \simeq S^t$

$$\pi_{t,w}X \simeq \pi_t \tau^{-1}X \simeq \mathrm{colim}_k \pi_{t,k}X.$$

- More on $\mathrm{Mod}_{C\tau}$.

Hovey's stable ∞-cat of comodules Stable_Γ.

$D(\mathrm{Comod}_\Gamma)$ bad [Hovey 03]

Instead, he constructed a different model str on $\mathrm{Ch}(\mathrm{Comod}_\Gamma)$ w.e. = htpy iso. (resp. hom iso)

$$\xrightarrow{\text{(stable) htpy cat}} D(\cdot\cdot) =: \mathrm{Stable}_\Gamma .$$

Recall

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{C}) \arrow[d, "\text{htpy cat}"] \arrow[dr] & \\
K(\mathcal{C}) \arrow[d, "\text{Verdier quo}"] & \mathrm{Ch}(\mathcal{C})/\{\text{acyclic}\} \arrow[dl, "\text{htpy cat.}"] \\
D(\mathcal{C}) &
\end{tikzcd}$$

FACT 1) Stable_Γ has all desired prop in $D(\mathrm{Comod}_\Gamma)$.

2) ·· will recover E_2-page of ASS.

$$\mathrm{Stable}_\Gamma(S^0A, S^0M)_* \cong \mathrm{Ext}^*_\Gamma(A, M)$$

where (A, Γ) Hopf algebroid.

S^0- : trivial cpx concentrated at deg 0.

3) (Piotr, Thm 3.7)

$$\mathrm{Stable}_\Gamma \simeq \underbrace{\mathrm{Shv}^{\mathrm{Sp}}_\Sigma}_{\substack{\text{product-pre.}\\ \text{sheaf valued}\\ \text{in Sp}}}(\underbrace{\mathrm{Comod}^{fp}_\Gamma}_{\text{dualizable }\Gamma\text{-comod.}}).$$

Again, assume $E \geq E_1$, Adams type. $\Gamma = E_*E$.

$E_* : \mathrm{Sp}^{fp}_E \longrightarrow \mathrm{Comod}^{fp}_\Gamma$.

$\mathrm{Shv}^{\mathrm{Sp}}_\Sigma(\mathrm{Sp}^{fp}_E)$ $\quad\quad$ $\mathrm{Shv}^{\mathrm{Sp}}_\Sigma(\mathrm{Comod}^{fp}_\Gamma)$.

Lem (4.43) $$\begin{tikzcd} \varepsilon^* : \mathrm{Syn}_E \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Stable}_\Gamma : \varepsilon_* \arrow[l, shift left] \end{tikzcd}$$

Here $\varepsilon^* : \nu X \longmapsto \nu(E_* X)$

or $\varepsilon^* =$ sheafification $\circ \operatorname{Lan}_{E_*}$.

$(\varepsilon_* X)(P) \simeq X(E_* P)$. or $\varepsilon_* =$ precomposition.

Induced by E_*.

FACT ε^* strict sym mon.
ε_* lax " " $\Big\}$ E E_∞.
otherwise. drop "sym".

Lem $P \in \mathrm{Sp}_E^{fp}$. $C\tau \otimes \nu P \simeq \varepsilon_*(E_* P)$.

Thm (4.46) $\exists$ adjoint pair

$$\begin{tikzcd} \mathcal{X}_* : \mathrm{Stable}_\Gamma \arrow[r, shift left] \arrow[r, phantom, "\top"] & \mathrm{Mod}_{C\tau}(\mathrm{Syn}_E) : \mathcal{X}^* \arrow[l, shift left] \end{tikzcd}$$

& $\mathcal{X}_*$. $\mathcal{X}^*$ $\overset{E_\infty}{(\text{sym})}$ mon.

Note that $\mathbb{1}_{\mathrm{Stable}_\Gamma} = E_*$ $(\Gamma = E_* E)$.

$$\begin{tikzcd} \mathrm{Stable}_\Gamma \arrow[rr, "\mathcal{X}_*"] \arrow[dr, "\varepsilon_*"'] & & \mathrm{Mod}_{C\tau}(\mathrm{Syn}_E) \arrow[dl] \\ & \mathrm{Syn}_E & \end{tikzcd}$$

$\varepsilon_* E_* \simeq C\tau$

1) $\mathcal{X}_*(E_* P) = C\tau \otimes \nu P$

2) $\mathcal{X}^*(C\tau \otimes \nu P) = \varepsilon^*(\nu P) = E_* P$.

Prop $\mathcal{H}^*$, $\mathcal{H}_*$ equiv. if TFAE:

① E Landweber exact

② E has enough projectives, i.e.

$\{E_*P : P \in Sp_E^{fp}\}$ gen. Stable_Γ through colim & Σ.

3. Applications.

$X, Y \in Sp$. Thm + Prop tells us (as a corollary)

$$[\nu Y, C\tau \otimes \nu X]_{t,w} \simeq \mathrm{Ext}_{E_*E}^{w-t,w}(E_*Y, E_*X)$$

Notation $[X, Y]_{t,w} = [S^{t,w} \otimes X, Y] = Y^{-t,-w}(X)$.

Replace Y by $\mathbb{S}$. then

$$\begin{aligned} & [\nu S^0 \otimes S^{t,w}, C\tau \otimes \nu X] \\ = & [S^{t,w}, C\tau \otimes \nu X] \\ = & \pi_{t,w}(C\tau \otimes \nu X) \\ \cong & \mathrm{Ext}_{E_*E}^{w-t,w}(E_*, E_*X). \end{aligned}$$

$$\begin{tikzcd}
\pi_{t,w}(C\tau \otimes \nu X) \arrow[r, phantom, "\cong"] & \mathrm{Ext}_{E_*E}^{w-t,w}(E_*, E_*X). \\
E_2 \ \tau\text{-Bockstein SS} \arrow[u, squiggly] & E_2 \ E\text{-based Adams SS.} \arrow[u, squiggly]
\end{tikzcd}$$

Can compute Adams SS by τ-inversion. "killing τ".

- Relation to "$C\tau$ - method":

Thm (Gheorghe - Wang - Xu)

$$\mathrm{Mod}^{\mathrm{cell}}_{C\tau} \simeq D(\mathrm{Comod}_{BP_*BP})$$

$$\text{motivic ASS} \cong \text{algebraic ANSS.}$$

Thm (Piotr 7.34)

After p-completion.

Chow-Novikov str.

$$\begin{tikzcd}
 & \mathrm{Mod}_{C\tau}(\mathrm{Syn}^{\mathrm{ev}}_{MU}) \arrow[dl, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & \\
\mathrm{Mod}_{C\tau}(\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell.\ harmonic}}) & & \mathrm{Stable}^{\mathrm{ev}}_{MU_*MU}
\end{tikzcd}$$

motivic cellular spectra / Spec $\mathbb{C}$.

Consequence : motivic $C\tau$ = special case of sth. in Syn_E.

where E = MU . BP . . . .